Spectral corrections for a class of eigenvalue problems

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Abstract. Consider eigenvalue problems of the form

\[ D_\lambda y = 0, \quad y(x) \in [a, b], \]
\[ L_a[y; \lambda] = L_b[y; \lambda] = 0, \]

where \( D_\lambda \) is a second order linear differential operator and \( \{L_a, L_b\} \) are linear functionals defining the boundary conditions at \( a \) and \( b \). Solutions to problem (1)-(2) consist of finding values for \( \lambda \), called eigenvalues, and their corresponding nontrivial solutions \( y(\lambda; x) \) which are called eigenfunctions.

There is a wide class of numerical methods that approximate the eigenvalues of (1)-(2). Once an approximation \( \tilde{\lambda} \) for the exact eigenvalue \( \lambda \) is computed with a certain method, a natural question that arises is how much does \( \tilde{\lambda} \) deviate from the (unknown) \( \lambda \). In the present work we are concerned with deriving a practical formula that allows to measure the difference \( \delta := \lambda - \tilde{\lambda} \) when \( \tilde{\lambda} \) is obtained with an exponentially fitted version of Legendre Gauss tau method. To achieve this we construct first an approximate eigenfunction \( \tilde{y} \) that satisfies condition \( L_a[\tilde{y}; \lambda] = 0 \), and then, using a new approach developed in [7], we introduce the miss-distance function \( F(\lambda) := L_b[y; \lambda] - L_b[\tilde{y}; \tilde{\lambda}] \) that measures the extent to which the second boundary condition \( L_b[y; \lambda] = 0 \) is satisfied.

Considering the Taylor’s series expansions of \( F(\lambda) \) near \( \tilde{\lambda} \),

\[ F(\lambda) = \sum_{k=0}^{\infty} \frac{\partial^k F}{\partial \lambda^k} \bigg|_{\lambda=\tilde{\lambda}} \delta^k, \]

we find that some estimates \( \tilde{\delta} \) for \( \delta \) can be generated by truncating the series above after a certain degree \( n \). Adding this estimate to \( \tilde{\lambda} \) we obtain the corrected eigenvalue \( \lambda^* := \tilde{\lambda} + \tilde{\delta} \). Numerical examples illustrating the accuracy of the proposed error estimates are provided.

1 Introduction

The study of many physical phenomena, such as the vibration of strings, the interaction of atomic particles, or the earth’s free oscillations yields Sturm-Liouville (SL) eigenvalue problems. The general form of SL problems that concerns this paper is

\[ -(p(x)y'(x))' + q(x)y(x) = \lambda r(x)y(x), \quad x \in [a, b], \]
\[ L_1[y] := c_1y(a) + p(a)y'(a) = g_1(\lambda), \quad L_2[y] := c_2y(b) - p(b)y'(b) = g_2(\lambda), \]

where \(-\infty < a < b < \infty, c_1, c_2 > 0, \{p(x), q(x), r(x)\} \) are continuous functions on \([a, b], p > 0, q > 0 \) and \( r \geq 0 \) on \([a, b], \lambda \) is an unknown parameter, and \( g_1, \)
$g_2$ are $\lambda$-dependent functions. Solutions to SL problem (3)-(4) consist of finding values for $\lambda$, called \textit{eigenvalues}, and their corresponding nontrivial solutions $y(\lambda; x)$ which are called \textit{eigenfunctions}.

There is a well-developed theory for SL problems (see Zettl [21], Pryce [18]): there is an infinite number of eigenvalues $\{\lambda_k\}_{k=0}^{\infty}$ for (3)-(4), which are real, and can be ordered such that $0 < \lambda_0 < \lambda_1 < ... < \lambda_k \to \infty$; for each $\lambda_k$, the corresponding eigenfunction oscillates and has $k - 1$ zeros in the open interval $(a,b)$. Thus for large $k$, an eigenfunction is highly oscillatory.

There is a wide class of numerical approaches, reported in the literature, that approximate the eigenvalues of (3)-(4): One can convert it to a first order system and then use standard numerical methods such as Runge-Kutta methods and collocation (see Ascher \textit{et al} [1]). Alternatively, one can apply conventional numerical techniques for second order differential equations (ODE) such as Numerov, Runge-Kutta-Nystrom or de Vogelaere methods combined with shooting techniques (see Ghelardoni and Gheri [6], Lambert [12], Vanden Berghe and De Meyer [19] and the references therein). Other well-studied methods that can be used are those with variational characterisation that include Rayleigh-Ritz.

Some of those approaches can be satisfactory in the case where only the first few eigenvalues are desired. But its usefulness becomes questionable and the accurate computation of eigenvalues tends to be a challenging problem in the cases where one wishes to compute a large number of eigenvalues. A famous example of this is the radial Schrödinger equation

$$
y'' + (\lambda - V(x))y = 0,
y(-\infty) = y(\infty) = 0,
$$

with $V(x)$ being a known potential function, which arises in quantum mechanics in the study of atom-ion interaction. For this equation, more than ten eigenvalues are usually sought, and the corresponding eigenfunctions become increasingly oscillatory. Due to the unsatisfactory performance of standard methods in detecting these strong oscillations, efforts have concentrated on developing modern techniques. Among those techniques, that have proven to be highly accurate and effective, are methods based on \textit{coefficients perturbation} such as CP-method or LP-method (see Ixaru [11], Ledoux \textit{et al} [16]) or Legendre-Gauss Tau method (see Canuto \textit{et al} [2], Gottlieb and Orszag [9], Ortiz [17], Lanczos [13]).

In a recent paper, El-Daou and Al Matar [4] proposed an exponentially weighted version of Legendre Gauss Tau method (ELGT). This method has shown its superiority in approximating ODEs with highly oscillatory solutions.

A natural question arises when a eigenvalue $\lambda$ is approximated by means of a numerical method is how much does the computed value $\tilde{\lambda}$ deviate from the (unknown) exact $\lambda$. In this paper we are concerned with measuring the
difference $\delta := \lambda - \tilde{\lambda}$ between the exact SL eigenvalue $\lambda$ and its approximate value $\tilde{\lambda}$ that is obtained with ELGT. We will give a formula for evaluating $\delta$ that allows to correct and improve $\tilde{\lambda}$ substantially.

2 Exponentially weighted Legendre Gauss Tau method

Let us consider the initial value problem (IVP),

$$
Dy := y'' + p_1(x)y' + p_0(x)y = f(x), \ x \in [a, b],
$$

$$
y(a) = \alpha_0, \ y'(a) = \alpha_1
$$

where $p_0(x)$ and $p_1(x)$ are continuous functions in $[a, b]$.

Legendre-Gauss Tau method of order $N$, (LGT($N$)), seeks an approximation $y_N$ for $y$ written as

$$
y_N = \sum_{i=0}^{N+1} a_i L_i(x),
$$

where $L_i(x)$ stands for Legendre polynomial of degree $i$ shifted to interval $[a, b]$, and where the unknown coefficients $\{a_i; i = 0, 1, \ldots, N + 1\}$ are determined by

(i) imposing the supplementary conditions on $y_N$,

$$
y_N^{(k)}(a) = \alpha_k; \ k = 0, 1
$$

(ii) and by forcing the residual $R_N(x) := Dy_N(x) - f(x)$ to vanish at the $N$ zeros of $L_N(x)$, $\{z_i\}_{i=1}^N \subset I$, (known as Legendre-Gauss points),

$$
R_N(z_i) = 0, \ i = 1, 2, \ldots, N.
$$

In the piecewise version of LGT we consider a partition $a = x_0 < x_1 < \ldots < x_M = b$ of $[a, b]$; $h_i = x_i - x_{i-1}$, and we use LGT($N$) to solve the $M$ IVPs,

$$
(Dy_i)(x) = f(x), \ x \in [x_{i-1}, x_i], \ i = 1, 2, \ldots, M,
$$

$$
y_i^{(k)}(x_{i-1}) = y_{i-1}^{(k)}(x_{i-1}), \ y_i^{(k)}(x_0) = \alpha_k, \ k = 0, 1.
$$

Throughout the paper, LGT($M,N$) will refer to piecewise LGT that uses $N$ Legendre-Gauss points on each one of the $M$ subintervals $\{[x_{i-1}, x_i], i = 1, 2, \ldots, M\}$. When $M = 0$, LGT(0,N)≡LGT(N).

The Exponentially Weighted Legendre (ELGT), needs to associate to the operator

$$
(Du)(x) = u'' + p_1(x)u' + p_0(x)u
$$

an auxiliary differential operator $D_\omega$ defined as

$$
D_\omega u := u'' + [2\omega + p_1(x)]u' + [\omega^2 + p_1(x)\omega + p_0(x)]u. \tag{5}
$$
where $\omega$ is a real or complex number. We proved in [4] that the exact solution of homogenous second order IVP
\[ Dy := y'' + p_1(x)y' + p_0(x)y = 0, \quad x \in [a, b], \]
\[ y(a) = \alpha_0, \quad y'(a) = \alpha_1 \]
is expressible as a linear combination of $\{e^{\omega_1x}, e^{\omega_2x}\}$,
\[ y = \phi_1(x)e^{\omega_1x} + \phi_2(x)e^{\omega_2x}, \]
where the frequencies $\{\omega_1, \omega_2\}$ are the (real or complex) roots of the quadratic equation
\[ \omega^2 + p_1(\bar{X})\omega + p_0(\bar{X}) = 0, \quad \bar{X} = X + h/2, \]
and where $\{\phi_1(x), \phi_2(x)\}$ are exact solutions of
\[ (D_{\omega_j}\phi_j)(x) := \phi''_j + (2\omega_j + p_1(x))\phi'_j + (\omega_j\delta p_1 + \delta \phi_0)\phi = 0. \tag{7} \]

It follows then that the exact solution of any homogenous second-order linear ODE can be represented in terms of variable amplitudes $\{\phi_1(x), \phi_2(x)\}$ and oscillatory (or hyperbolic) weights $\{e^{\omega_1x}, e^{\omega_2x}\}$. It turns out that in order to obtain the exact solution $y(x)$, we need to find the amplitudes $\{\phi_1(x), \phi_2(x)\}$, and then determine the constants $\{c_1, c_2\}$ for $y = c_1\phi_1(x)e^{\omega_1x} + c_2\phi_2(x)e^{\omega_2x}$ in a way that the given initial conditions are satisfied.

Analytically, solving (7) is not easier, however, than solving the original problem (6). But, computationally, since the oscillatory factor is taken out, numerical methods that approximate the smooth solutions of (7) could be more successful than approximating (6) directly, specially when $y(x)$ exhibits sharp variations. Next we will propose a procedure for LGT that can effectively generate approximations $\{\tilde{\phi}_1(x), \tilde{\phi}_2(x)\}$ for the functions $\{\phi_1(x), \phi_2(x)\}$ defined by (7) and then construct an approximation $\tilde{y} = c_1\tilde{\phi}_1e^{\omega_1x} + c_2\tilde{\phi}_2e^{\omega_2x}$ for $y$. We will refer to this procedure by ELGT(M,N) where $M$ indicates the number of steps and $N$ is the number of Legendre-Gauss points used in each subinterval $[X, X + h]$. The following algorithm lists the steps of ELGT(M,N) procedure for approximating the IVP
\[ y'' + p_1(x)y' + p_0(x)y(x) = 0, \quad x \in [a, b], \]
\[ y(a) = \alpha_0, \quad y'(a) = \alpha_1 \]

**ELGT-Procedure:**

1. Define a partition $a = x_0 < x_1 < \ldots < x_M = b$ of $[a, b]$; set $h_i = x_i - x_{i-1}$.
2. Provide $\{z_k\}_{k=1}^N$, the $N$ Legendre-Gauss points in $[0,1]$. 

3. For $i = 1, 2, \ldots, M$ repeat (a)-(d):
(a) Compute $\{\omega_{1i}, \omega_{2i}\}$ for $[x_{i-1}, x_i]$:
\[
\omega_1 = \frac{1}{2} \left(-p_1(\bar{x}_i) + \sqrt{p_1(\bar{x}_i)^2 - 4p_0(\bar{x}_i)}\right), \quad \omega_2 = \frac{1}{2} \left(-p_1(\bar{x}_i) - \sqrt{p_1(\bar{x}_i)^2 - 4p_0(\bar{x}_i)}\right), \quad \bar{x}_i = x_{i-1} + \frac{h_i}{2}.
\]
(b) Obtain an LGT(M,N) approximation $\phi_{N,i,1} = \sum_{j=0}^{N} a_{ji} L_{ji}(x)$ where $\{a_{ji}\}$ satisfy the linear system,
\[
(D_{\omega_1} \phi_{N,i,1})(x_{i-1} + h_i z_k) = 0, \quad \phi_{N,i,1}(x_{i-1}) = 1, \quad k = 1, 2, \ldots, N.
\]
(c) Obtain an LGT(M,N) approximation $\phi_{N,i,2} = \sum_{j=0}^{N} b_{ji} L_{ji}(x)$ where $\{b_{ji}\}$ satisfy the linear system,
\[
(D_{\omega_2} \phi_{N,i,2})(x_{i-1} + h_i z_k) = 0, \quad \phi_{N,i,2}(x_{i-1}) = -1, \quad k = 1, 2, \ldots, N.
\]
(d) Construct ELGT(M,N) approximation $y_{N,i} = c_{1i} \phi_{N,i,1} e^{\omega_{1i} x} + c_{2i} \phi_{N,i,2} e^{\omega_{2i} x}$ where $\{c_{1i}, c_{2i}\}$ are fixed by left-end conditions
\[
y_{N,i}^{(\ell)}(x_{i-1}) = y_{N,i-1}^{(\ell)}(x_{i-1}), \quad \ell = 0, 1.
\]

3 Sturm-Liouville problems

In this section we consider the numerical solution of regular SL eigenvalues problems (3)-(4) by means of the exponentially weighted Legendre-Gauss Tau method. Problem (3)-(4) is called regular if $a$ and $b$ are finite, (see Pryce [18] and Zettl [21]). As described in the previous section, ELGT is designed for treating initial value problems. But in order to solve SL eigenvalue problems we need to combine it with some shooting technique as explained next.

Let us associate to (3)-(4) the IVP
\[
-(p(x)u'(x))' + q(x)u(x) = \lambda r(x)u(x), \quad x \in [a, b],
\]
(8)
\[
L_1[u] = g_1(\lambda), \quad u'(a) = 1.
\]
(9)

Let us assume that $Y(\lambda, h, x)$ is an approximation for $u(x)$ obtained with ELGT(M,N) for some prescribed $M$ and $N$ where $h := \max\{h_i, \ i = 1, 2, \ldots, M\}$. Then we define the miss-distance function $F(\lambda, h)$ by
\[
F(\lambda, h) = L_2[Y] - g_2(\lambda) := c_2 Y(\lambda, h, b) - p(b)Y'(\lambda, h, b) - g_2(\lambda).
\]
(10)
\( F(\lambda, h) \) describes the extent to which the boundary condition defined by the linear functional \( L_2 \) at the right end point \( b \) is satisfied. For each fixed \( h \), we consider \( \tilde{\lambda} := \hat{\lambda}(h) \) as an approximation to the eigenvalue \( \lambda \) whenever \( \hat{\lambda} \) is a zero of the miss-distance (10); that is
\[
F(\tilde{\lambda}, h) = 0. \tag{11}
\]

The following algorithm lists the steps necessary for generating the \( \tilde{\lambda}'s \):

\textit{Algorithm ELGT-SL}

1. Define \( M \), the number of partition points of \([a, b]\)
2. Define \( N \), the number of Gauss points in \([0,1]\).
3. Form the partition \( a = x_0 < x_1 < \ldots < x_M = b \), with \( h_i = x_i - x_{i-1} \).
4. Solve IVP (8)-(9) by ELGT(M,N)
5. Construct the miss-distance function (10)
6. Solve (11) to obtain \( \{\tilde{\lambda}\} \).

4 Correcting SL eigenvalues

A natural question arises when a certain quantity is approximated by means of a numerical method is how much does the computed value deviate from the (unknown) exact value. In this section we are concerned with measuring the difference \( \delta := \lambda - \hat{\lambda} \) between the exact SL eigenvalue \( \lambda \) and its approximate value \( \hat{\lambda} \) that is obtained with ELGT(M,N). We will give a formula for evaluating \( \delta \) that allows to correct and improve \( \hat{\lambda} \) substantially.

Our next result follows from an approach used in Ghelardoni \textit{et al} [7].

\textbf{Theorem 1} Let us assume that \( F(s, t) \) defined in (10) is continuous in both variables \( s \) and \( t \), and sufficiently differentiable with respect to \( s \). Let \( \hat{\lambda} \) be an approximation of the eigenvalue \( \lambda \) obtained by ELGT(M,N) for given \( M = (b-a)/h \) and \( N \). Then the difference \( \delta := \lambda - \hat{\lambda} \) satisfies the following formal infinite series
\[
\sum_{k=0}^{\infty} F^{(k)}(\tilde{\lambda}, 0) \frac{\delta^k}{k!} = 0 \tag{12}
\]
where \( F^{(k)}(\tilde{\lambda}, 0) \) stands for the \( k \)th partial derivative of \( F(\lambda, 0) \) with respect to \( \lambda \) evaluated at \( \lambda = \tilde{\lambda} \).

\textbf{Proof.} Since \( F \) is sufficiently differentiable with respect to \( s \), we can expand \( F \) near \( \hat{\lambda} \) as
\[
F(s, 0) = \sum_{k=0}^{\infty} F^{(k)}(\tilde{\lambda}, 0) \frac{(s - \hat{\lambda})^k}{k!} \tag{13}
\]
where \( F^{(k)}(s, t) := \frac{\partial^k F}{\partial s^k}(s, t) \). In particular, for \( s = \lambda \), we have

\[
F(\lambda, 0) = \sum_{k \geq 0} F^{(k)}(\tilde{\lambda}, 0) \frac{\delta^k}{k!}.
\]

Due to the continuity of \( F \)

\[
\lim_{h \to 0} F(\mu_k, h) = F(\lambda, 0) = L_2[y(\lambda, b)] = 0.
\]

This, combined with (13), gives (12).

From expansions (12) one can generate linear and quadratic estimations for \( \delta \):

(i) Linear estimation \( \delta_1 \) is obtained if we truncate (12) after \( k = 1 \);

\[
F(\tilde{\lambda}, 0) + F'(\tilde{\lambda}, 0) \delta = 0.
\]

Hence

\[
\delta_1 := \delta = -\frac{F(\tilde{\lambda}, 0)}{F'(\lambda, 0)}.
\]

(ii) Quadratic estimation \( \delta_2 \) is obtained if (12) is truncated after \( k = 2 \):

\[
F(\tilde{\lambda}, 0) + F'(\tilde{\lambda}, 0) \delta + \frac{1}{2} F''(\tilde{\lambda}, 0) \delta^2 = 0
\]

and therefore

\[
\delta_2 := \delta = \min\{\frac{-F'(\tilde{\lambda}, 0) \pm \sqrt{(F'(\tilde{\lambda}, 0))^2 - 2F''(\tilde{\lambda}, 0)F(\tilde{\lambda}, 0)}}{F''(\lambda, 0)}\}.
\]

The corrected eigenvalues obtained from \( \tilde{\lambda} \) by adding \( \delta_1 \) and \( \delta_2 \) will be respectively denoted by

\[
\lambda^* = \tilde{\lambda} + \delta_1 \quad \text{and} \quad \lambda^{**} = \tilde{\lambda} + \delta_2.
\]

The following algorithm describes the correction procedure:

1. Compute \( \tilde{\lambda} \) with ELGT(M,N)
2. Take a sufficiently large real \( \alpha \)
3. Set \( \tau = 10^{-\alpha} \)
4. Estimate \( F(\tilde{\lambda}, 0) \) by \( F(\tilde{\lambda}, \tau) \)
5. Estimate \( F'(\tilde{\lambda}, 0) \) by \( F'(\tilde{\lambda}, \tau) \)
6. Estimate \( F''(\tilde{\lambda}, 0) \) by \( F''(\tilde{\lambda}, \tau) \)
7. Compute \( \Delta_\pm \) as

\[
\Delta_\pm = \frac{-F'(\tilde{\lambda}, 0) \pm \sqrt{(F'(\tilde{\lambda}, 0))^2 - 2F''(\tilde{\lambda}, 0)F(\tilde{\lambda}, 0)}}{F''(\lambda, 0)}
\]

8. Compute \( \delta_1 = -\frac{F(\tilde{\lambda}, 0)}{F'(\lambda, 0)} \)
9. Compute \( \delta_2 = \min\{\Delta_\pm\} \)
10. Compute \( \lambda^* = \tilde{\lambda} + \delta_1 \)
11. Compute \( \lambda^{**} = \tilde{\lambda} + \delta_2 \)
5 Numerical examples

Next we illustrate numerically the sharpness of the correction procedure presented above:

**Example 1** Consider the SLP

\[ y'' + (\lambda - \exp(x))y = 0 \quad x \in [0, \pi]; \quad (14) \]
\[ y(0) = y(\pi) = 0. \quad (15) \]

We obtained approximations \{\lambda_k\} for \{\lambda\} by applying ELGT(M,N) with \(M = 10\) and \(N = 2\) and 4. These \lambda_k’s were corrected using our procedure in both cases: linear and quadratic. Our results are listed in Table 1 compared to those given in [8].

**Table 1.** Absolute errors in the eigenvalues of (14)-(15) obtained with ELGT(M,N) and their corrected values are displayed. A comparison with results reported in [8] is given in the last column

<table>
<thead>
<tr>
<th>index</th>
<th>Exact e.v</th>
<th>ELGT(10,2)</th>
<th>ELGT(10,4)</th>
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</table>

**Example 2** Consider now the SLP (see [16])

\[ y'' + (E - V(x))y = 0, \quad [0, x_r], \quad (16) \]

with the boundary conditions

\[ y(0) = 0 \quad \text{and} \quad (\sqrt{V(x_r) - E})y(x_r) + y'(x_r) = 0, \quad (17) \]

where \(E\) is the energy to be determined and \(V(x)\) is the Woods-Saxon potential

\[ V(x) = v_0 W(x) \left[ 1 - \frac{1 - W(x)}{a_0} \right], \]
\[ W(x) = \left[ 1 + \exp\left( \frac{x - x_0}{a_0} \right) \right]^{-1}, \]

\[ v_0 = -50, \quad x_0 = 7, \quad a_0 = 0.6 \]

and \( x_r = 15 \). In fact this choice for \( x_r \) is due to the behavior of \( V(x) \) in the region \( x > 15 \). From Figure 5 (right curve), we see that since \( V(x) \) is negligible for \( x > 15 \), we can safely take \( x_r = 15 \) and therefore integrate problem (16)-(17) in the interval \([0, 15]\).

We have applied ELGT(M,4) with \( M = 15, \) and 30 ; (that is \( h = 1, \) and \( \frac{1}{2} \)). Table 2 (Col 3) displays the errors committed by ELGT.

Let us consider correcting these eigenvalues by using the linear correction and quadratic correction. In Table 2 (Cols 4 and 5), we give the absolute errors for LGT(15,4) (upper rows) and for ELGT(30,4) (lower rows).

In Figure 5 (left curve) we plot the eigenfunction that corresponds to \( \tilde{E}_{13} = -3.908234316753899 \).

**Example 3** Here we wish to solve SLP (16)

\[ y'' + (E - V(x))y = 0, \quad x \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right], \]

with \( V(x) \) being the potential function of Coffey-Evans (see [18]):

\[ V(x) = -2\beta \cos 2x + \beta^2 \sin^2 2x, \quad \beta = 30 \]

in interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\) and where conditions (17) are replaced by

\[ y\left( -\frac{\pi}{2} \right) = y\left( \frac{\pi}{2} \right) = 0. \]

For this problem, \( \lambda_0 \) is very close to zero and there are very close eigenvalues triplets \( \{\lambda_2, \lambda_3, \lambda_4\}, \{\lambda_6, \lambda_7, \lambda_8\}, \ldots \), with the other eigenvalues well separated.

The approximate eigenvalues \( \{\tilde{\lambda}_k\} \) of this problem were obtained by means of ELGT(M=75,N=4) (see Table 3). The high accuracy of our method is reflected in the detection of close triplets \( \{\tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4\}, \{\tilde{\lambda}_6, \tilde{\lambda}_7, \tilde{\lambda}_8\}, \ldots \) and \( \{\tilde{\lambda}_{10}, \tilde{\lambda}_{11}, \tilde{\lambda}_{12}\} \).

**Example 4** Finally let us consider the \( \lambda \)-rational SLP (see [7])

\[ y'' + (\lambda + \frac{q(x)}{u(x) - \lambda})y = 0, \quad x \in [a, b], \quad (18) \]

where \( q(x) = -\exp(-x) \) and \( u(x) = 1 - \exp(-x) \), associated with the boundary conditions

\[ u'(a)y(a) - u(a)y'(a) = 0 \quad \text{and} \quad u'(b)y(b) - u(b)y'(b) = 0. \quad (19) \]
Fig. 1. Plot of the eigenfunction that corresponds to $\hat{E}_{13} = -3.90823244316753899$ for example 2 (left). Plot of the potential $V(x)$ in interval $[0,15]$ (right)

Table 2. Absolute errors in the eigenvalues of (16)-(17) obtained with ELGT(M,N) and corrected using 1st and 2nd order corrections. For each $k$, in upper row $M=15$, $N=4$; in lower row $M=30$, $N=4$

<table>
<thead>
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<th>$\lambda_k - \lambda_0^*$</th>
<th>$\lambda_k - \lambda_0^{**}$</th>
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<td>-2.1E-5</td>
<td>+4.0E-6</td>
<td>+2.7E-9</td>
</tr>
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<td>-2.5E-5</td>
<td>-6.1E-6</td>
<td>-5.5E-9</td>
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<td>-1.3E-12</td>
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<td>+2.4E-3</td>
<td>+1.2E-5</td>
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</table>
Table 3. Errors in the eigenvalues for Coeffy-Evans problem obtained with ELGT(75,4) and corrected 2nd order corrections.

<table>
<thead>
<tr>
<th>k</th>
<th>$\lambda_k$</th>
<th>$\tilde{\lambda}_k$</th>
<th>$\bar{\lambda}_k$</th>
<th>$\lambda_k - \lambda_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>-2.36035695339E-12</td>
<td>3.268E-10</td>
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<tr>
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</tr>
<tr>
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<td>13</td>
<td>637.7265023255106</td>
<td>637.7265023185399</td>
<td>-6.999E-9</td>
<td></td>
</tr>
</tbody>
</table>
The essential spectrum of this type of SLP is the interval \([u(a), u(b)]\). Finding eigenvalues outside the essential spectrum is straightforward, so we wish to find \(\lambda \in [u(a), u(b)]\). Spectral problems of this type arise as Hain-Lust equations in some questions of magnetohydrodynamics (see [14], [15]). In this case \(\lambda\) is called eigenvalue embedded in the essential spectrum.

In our example, for \(a = -5\) and \(b = 4\), there is the embedded eigenvalue \(\lambda = 0\) corresponding to the eigenfunction \(y(x) = 1 - \exp(-x)\).

We found that the classical Legendre-Gauss tau method gives satisfactory results. We have applied LGT(M,N) with M=10, 20, 40 and N=4; (that is \(h = 1, \frac{1}{2}\) and \(\frac{1}{4}\)). Table 4 displays some numerical results along those given in [7].

<table>
<thead>
<tr>
<th>(h)</th>
<th>(\hat{\lambda}_k)</th>
<th>(\frac{\lambda_k - \hat{\lambda}_k}{\lambda_k})</th>
<th>(\hat{\lambda}^*)</th>
<th>(\frac{\lambda^* - \hat{\lambda}^<em>}{\lambda^</em>})</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-6.28E-8</td>
<td>2.76E-14</td>
<td>1.00</td>
<td>1.89E-9</td>
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<tr>
<td>0.5</td>
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<td>0.25</td>
<td>-1.085E-12</td>
<td>1.84E-21</td>
<td>1.00</td>
<td>1.81E-15</td>
</tr>
</tbody>
</table>

Table 4. The embedded e.v. \(\lambda = 0\) for SLP (18)-(19) is obtained with LGT(M,N) and improved using linear correction. A comparison with results reported in [7] is given.

6 Conclusion

In this paper we introduced a practical formula to estimate the error when SL eigenvalues problems are approximated by means of Legendre Gauss Tau method in its classical and exponentially fitted versions. With such formula we can efficiently correct the computed eigenvalues. The correction procedure is tested on ELGT, and it can be used as well with other numerical methods provided a good initial approximation is supplied.

7 Acknowledgements

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References